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Time-of-Flight Analyses for the Gravitational Capture Maneuver

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Several aspects of the time of flight involved in the problem of gravitational capture in the regularized, restricted three-body problem are studied. A gravitational capture occurs when a massless particle changes its two-body energy around one celestial body from positive to negative without the use of nongravitational forces. The importance of several parameters involved in a capture in the Earth–moon system is studied. In particular, the time required for the capture is studied under different initial conditions. This information is very important for mission designers for two reasons: 1) the time for gravitational capture has an impact in the duration of any mission that uses this technique; and 2) the gravitational capture is a very sensitive phenomenon and, if the time involved is too large, perturbations from different sources may influence the mission too much. The results show the existence of regions where the time for the capture is reduced by a factor close to 10 with an adequate choice of initial conditions, without the reduction of the energy savings. In addition, the results are used to solve a family of optimization problems, such as the problem of finding trajectories that can perform a gravitational capture in a minimum time with a fixed savings or with maximum savings with an upper time limit.

Introduction

THE problem of gravitational capture in the regularized, restricted three-body problem is studied. For gravitational capture a phenomenon in which a massless particle changes its two-body energy around one of the primaries from positive to negative is understood. This capture is always temporary and, after some time, the two-body energy changes back to positive and the massless spacecraft leaves the neighborhood of the primary. Excellent studies of this problem are available in Refs. 1–3. The importance of this temporary capture is that it can be used to decrease the fuel expenditure for a mission going from one of the primaries to the other, as in a mission going from the Earth to the moon. The goal is to apply an impulse to the spacecraft during this temporary capture to accomplish a permanent capture. Because the goal of this impulse is to decrease the two-body energy of the spacecraft, its magnitude is smaller if it is applied during this temporary capture. An important application of this capture can be found in a special type of trajectory to the moon that consists of a sequence of maneuvers to send a spacecraft from the Earth to the moon with a fuel consumption smaller than the fuel required by the Hohmann transfer.[3–10] This maneuver is better described in the next sections.

The calculation of the time required to achieve a gravitational capture is emphasized in this paper. To perform this task, a large number of trajectories starting close to the secondary body are numerically integrated backward in time. The initial position and velocity of the spacecraft are changed, and it is verified if an escape occurs for every set of initial conditions. Remember that an escape in backward time is equivalent to a capture in forward time. Then, the time elapsed until the escape occurs is shown as a function of all of the parameters involved in this maneuver. From those results it is possible to study the balance between the time required for the capture and the energy saved for each maneuver.

Mathematical Model and Some Properties

The model used in all phases of this paper is the well-known planar, circular, restricted three-body problem. This model assumes that two main bodies M_1 and M_2 are orbiting their common center of mass in circular Keplerian orbits and a third body M_3, with negligible mass, is orbiting these two primaries. The motion of M_3 is supposed to stay in the plane of the motion of M_1 and M_2, and it is affected by both primaries, but it does not affect their motion.[11] The standard canonical system of units associated with this model is used. (The unit of distance is the distance between M_1 and M_2, and the unit of time is chosen such that the period of the motion of M_2 around M_1 is 2π.) Under this model, the equations of motion are

$$\ddot{x} - 2\dot{y} = x - \frac{\partial U}{\partial x} = \frac{\partial \Omega}{\partial x}, \qquad \ddot{y} + 2\dot{x} = y - \frac{\partial U}{\partial y} = \frac{\partial \Omega}{\partial y} \tag{1}$$

where Ω is the pseudopotential function given by

$$\Omega = \frac{1}{2}(x^2 + y^2) + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} \tag{2}$$

and x and y are two perpendicular axes with the origin in the center of mass of the system, with x pointing from M_1 (which has coordinates $x = -\mu$ and $y = 0$) to M_2 (which has coordinates $x = 1 - \mu$ and $y = 0$).

One of the most important reasons why the rotating frame is more suitable to describe the motion of M_3 in the three-body problem is the existence of an invariant, which is called the Jacobi integral (or energy integral). There are many ways to define the Jacobi integral and the reference system used to describe this problem (see Ref. 11, p. 449). In this paper the definitions used by Broucke[12] are followed. In this version, the Jacobi integral is given by

$$J = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - \Omega(x, y) = \text{const} \tag{3}$$

The equations of motion given by Eq. (1) are correct, but they are not suitable for numerical integration in trajectories passing near one of the primaries. The reason is that the positions of both primaries are singularities in the potential U (because r_1 or r_2 goes to zero or near zero) and the precision of the numerical integration is affected every time this situation occurs.

The solution for this problem is the use of regularization, which consists of a substitution of the variables for position $x - y$ and time t by another set of variables $(\omega_1, \omega_2, \tau)$, such that the singularities are eliminated in these new variables. Several transformations with this goal are available in the literature (Ref. 11, Chap. 3), such as those of Thiele-Burrau, Lemaître, and Birkhoff. They are called global regularization to emphasize that both singularities are eliminated at the same time. The case where only one singularity is eliminated at a time is called local regularization. For the present research, Lemaître's regularization is used.

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Gravitational Capture

To define the gravitational capture, it is necessary to use two basic concepts from two-body celestial mechanics. These concepts are as follows.

1) A spacecraft is in a closed orbit if its velocity is not large enough to escape from the central body. It remains always inside a sphere centered in the central body.

2) A spacecraft is in an open orbit if its velocity is large enough to escape from the central body. In this case the spacecraft can go to infinity, independent of its initial position.

To identify the type of orbit of the spacecraft, it is possible to use the definition of the two-body energy E of a massless particle orbiting a central body. The equation is $E = (V^2/2) - (\mu/r)$, where V is the velocity of the spacecraft relative to the central body in a nonrotating frame, μ is the gravitational parameter of the central body, and r is the distance between the spacecraft and the central body.

With this definition, it is possible to say that the spacecraft is in an open orbit if its energy is positive and that it is in a closed orbit if its energy is negative. In the two-body problem this energy remains constant, and it is necessary to apply an external force to change it. This energy is no longer constant in the restricted three-body problem. Then, for some initial conditions, a spacecraft can alternate the sign of its energy from positive to negative or from negative to positive. When the variation is from positive to negative the maneuver is called a gravitational capture, to emphasize that the spacecraft was captured by gravitational forces only, without the use of an external force, such as the thrust of an engine. The opposite situation, when the energy changes from negative to positive, is called a gravitational escape. In the circular, restricted three-body problem there is no permanent gravitational capture.[12–14] If the energy changes from positive to negative, it will change back to positive in the future. The mechanism of this capture is very well explained in Refs. 1–3.

Trajectories to the Moon Using Gravitational Capture

One of the most important applications of the gravitational capture can be found in trajectories to the moon.[3–10] The concept of gravitational capture is used together with the basic ideas of the gravity-assisted maneuver and the bielliptic transfer to generate a trajectory that requires a fuel consumption smaller than the one required by the Hohmann transfer. This maneuver consists of the following steps. 1) The spacecraft is launched from an initial circular orbit with radius r_0 to an elliptic orbit that crosses the moon's path. 2) A swing-by with the moon is used to increase the apoapsis of the elliptic orbit. This step completes the first part of the bielliptic transfer, with some savings in ΔV due to the energy gained from the swing-by. 3) With the spacecraft at the apoapsis, a second very small impulse is applied to raise the periapsis to the Earth–moon distance. Solar effects can reduce further the magnitude of this impulse. 4) The transfer is completed with the gravitational capture of the spacecraft by the moon. Using this technique, it is possible to obtain a practical result (savings in ΔV) from a temporary property of the dynamical system (the temporary gravitational capture).

Results

To quantify the gravitational captures, this problem is studied under several different initial conditions. The assumptions made for the numerical examples presented in the first part of this section are as follows. (Some of them are changed later, to generalize the results.)

1) The system of primaries used is the Earth–moon system. (Some fictitious systems are used later to generalize the results.)

2) The motion is planar everywhere because the out-of-plane capture cannot achieve larger savings.[2]

3) The starting point of each trajectory is 100 km from the surface of the moon ($r_p = 1838$ km from the center of the moon). Then, to specify the initial position completely, it is necessary to give the value of one more variable. The variable used here is the angle α, an angle measured from the Earth–moon line in the counterclockwise direction and starting on the side opposite to the Earth (Fig. 1). (Different values of r_p are used later to generalize the results.)

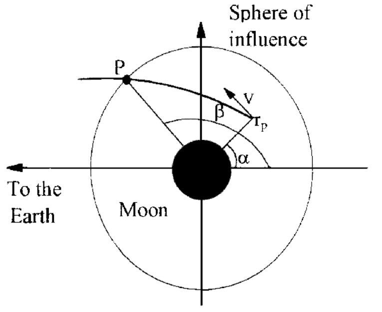


Fig. 1 Variables to specify the initial conditions of the spacecraft.

4) The magnitude of the initial velocity is calculated from a given value of $C3 = 2E = V^2 - (2\mu/r)$, where E is the two-body energy of the spacecraft with respect to the moon, V is the velocity of the spacecraft with respect to a nonrotating frame, μ is the gravitational parameter of the moon, and r is the distance between the spacecraft and the center of the moon. The direction of the velocity is assumed to be perpendicular to the line spacecraft–center of the moon and pointing in the counterclockwise direction for a direct (posigrade) orbit and in the clockwise direction for a retrograde orbit (Fig. 1).

5) To consider that an escape occurred, it is requested (following the conditions used in Refs. 1–3) that the spacecraft reach a distance of 100,000 km (0.26 canonical units) from the center of the moon in a time shorter than 50 days. The value for the distance comes from the equation for the limit $= (2\mu)^{1/3}$, which is well explained in Ref. 2. Figure 1 shows the point P where the escape occurs. The angle that specifies this point is called the entry position angle, and it is designated as β. There is also a check to verify that a crash into the moon did not happen.

Then, for each initial position, the trajectories were numerically integrated backward in time. Every escape in backward time corresponds to a gravitational capture in forward time. The time of flight until an escape occurs is obtained. The stopping criterion for the numerical integration is the one that comes first among the three possibilities: the time is longer than 50 days, the distance from the moon is greater than 100,000 km, or the distance from the moon is smaller than 1738 km (the moon radius). The numerical simulations were performed in an IBM-PC Pentium 100 MHz using the Microsoft Fortran Powerstation 1.0. The numerical integration method used is fourth-order Runge–Kutta. Then, the results are organized and plotted in several figures. The time of flight for escape in all of these figures is expressed in canonical units, which means that 1 unit of time is equivalent to 4.46 days. The next subsections detail the results.

Preliminary Analysis

This problem involves the restricted three-body problem, and it is well known in the literature that very few analytical results are available for this problem. The alternative that is available is the use of numerical simulations, combined with some analysis made with the two-body approximation.

To perform a preliminary analysis of this problem, the value of the minimum energy that allows an escape was calculated for every initial angle α. Plots of the minimum value of $C3$ vs α were made, and they confirm the results obtained previously by Yamakawa et al.[1] The results are not repeated here to save space. The most interesting conclusion is the existence of angles that provide maximum saving, such as the interval $150 \leq \alpha \leq 200$ deg, and that a direct orbit usually provides more savings than a retrograde orbit.

Then, the results available in the literature were generalized a bit, and the effects of the variation of the initial distance r_p from the moon were considered. Plots of the magnitude of the minimum $C3$ (radial variable) as a function of the departure angle α (angular variable) for several values of the initial distance were made. Figure 2 shows the results. To build Fig. 2, α was varied in steps of 5 deg. Figure 2 is separated into two parts: the first one shows the results when r_p is close to the surface of the moon (1838–21,838 km), and the second one shows the regions more distant from the surface of the moon (21,838–51,838 km). In the first plot is possible to see that, when r_p increases, $C3$ decreases. It also appears that the plots

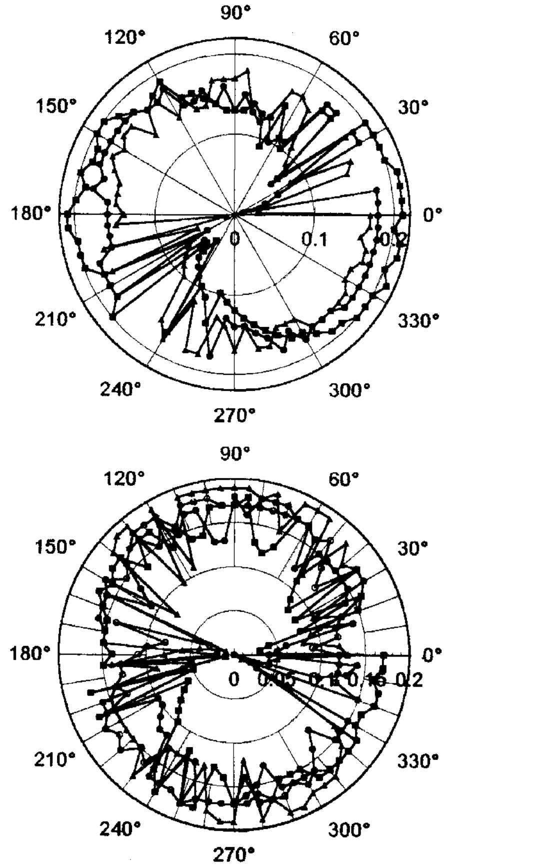


Fig. 2 Minimum $C3$ vs α for different values of r_p: top—■, 1838 km; ●, 11,738 km; and ▲, 21,738 km; and bottom—■, 21,738 km; ●, 31,738 km; ▲, 41,738 km; and ○, 51,738 km.

rotate in the clockwise direction when r_p increases. But, for the second diagram, when r_p is greater than 21,738 km the results are more confounding. Probably the influence of the Earth gravitational force disturbs the experiment. Therefore, there is no general rule to govern these effects.

Influence of the Parameters in the Time Required for Capture

In this section, the numerical tools developed are used to study the influence of the parameters that govern this problem. These parameters are the system of primaries involved, specified by the parameter μ; the distance from the spacecraft to the secondary body in the moment that the impulse is applied to complete the maneuver (r_p); the energy $C3$ of the spacecraft at this moment; the direction of the velocity at this point (it is assumed to be perpendicular to the radius vector, but it is free to be posigrade or retrograde); and the departure angle α (Fig. 1). The simulations showed the following results.

Effects of the Mass Parameter

Several simulations were made to study the influence of the mass parameter in the time required for the capture. This particular parameter brings a new difficulty into the problem. The value of the maximum savings increases greatly when the mass parameter increases. Thus, studying this problem for a fixed value of $C3 \approx -0.2$, that is, close to the maximum saving for $\mu = 0.01$, makes the time required for the capture to be close to zero for different values of μ ($\mu = 0.3, 0.5$, etc.). Then, the method used in this paper to solve this problem is to find and use a value of $C3$ that is close to the limit (minimum value that allows a gravitational capture) for a given μ. The comparison is made using a fixed value of $r_p = 0.004781477$ in canonical units, a posigrade direction for the velocity, and the value of $C3$ that is the one that gives the maximum saving for a given μ. The results showed that there is no general trend for the variation of this parameter. There are very large oscillations in the time required and, for every value of α, there is a different value of the mass parameter that holds the minimum time. This oscillatory behavior is due to the necessity of changing the values of $C3$, as explained in the beginning of the present section. The effect of this parameter is studied only for completeness, and it is not a key parameter for practical applications because in a real mission the system of primaries is always fixed in advance.

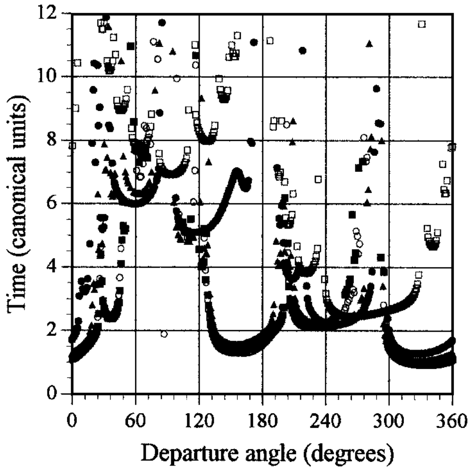


Fig. 3 Time of flight vs α for different values of r_p: ■, 1837 km; ●, 2738 km; ▲, 6738 km; □, 11,738 km; and ○, 21,738 km; $C3 = -0.17$.

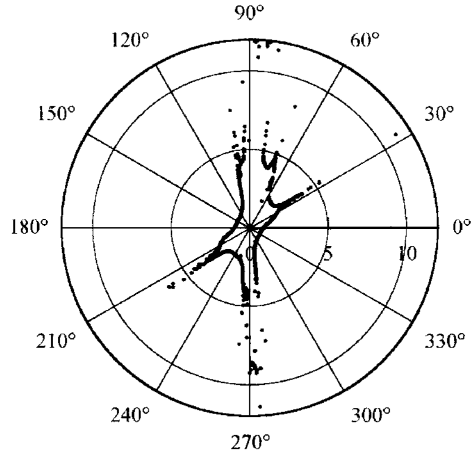


Fig. 4 Time of flight vs α for $C3 = -0.14$.

Effects of the r_p

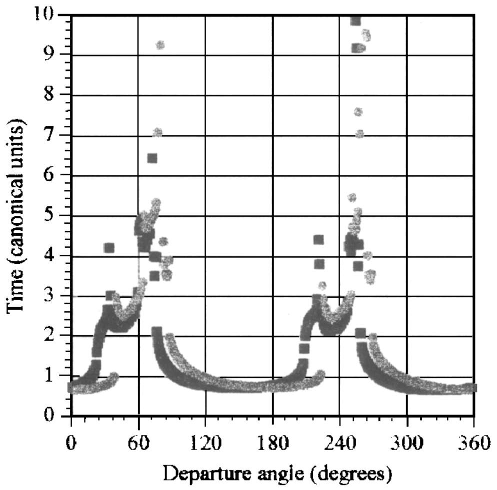
To study the importance of this parameter, simulations were made for the Earth–moon system ($\mu = 0.0121285627$) and for the posigrade direction of the velocity. The parameter r_p was varied over a wide range of values ($1838 \leq r_p \leq 21{,}738$ km) for several values of $C3$. Figure 3 shows the results obtained in the case $C3 = -0.17$. Simulations with a step of 1 deg in α were used. The first fact to be noted is that the results for $r_p = 1838, 2738$, and 6738 km are very similar to each other. This means that changing this parameter in a range of values close to the moon does not have a significant impact on the time required for the capture. There are exceptions in only a few points. Increasing this parameter to the values $r_p = 11{,}738$ and 21,738 km, it is possible to see an increase in the time of flight. The amount of this increase changes according to the value of α. In the maximum case, it reaches the level of three times larger than the value obtained with $r_p = 2738$ km. This situation occurs when α is between 50 and 200 deg. Thus, the general conclusion is that the increase of r_p has the effect of increasing the time for capture, but this effect is visible only for $r_p \geq 10{,}000$ km.

Effects of Departure Angle α

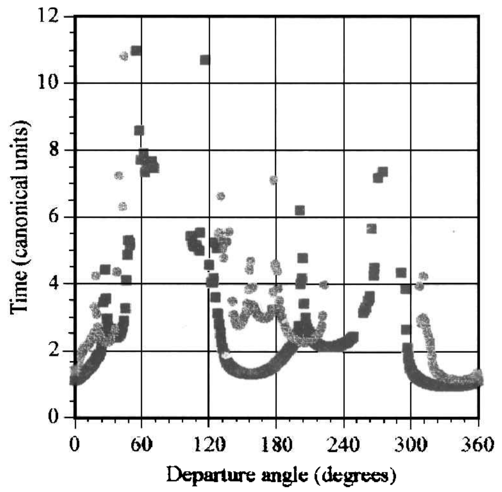
This is a very important parameter in this problem because it has a strong impact in the savings obtained for the maneuver. Simulations to measure the time of flight as a function of α were made for several values of $C3$ in the range $-0.2 \leq C3 \leq 0$. Figure 4 shows the results obtained for $C3 = -0.14$. Figure 4 was built using a step of 0.1 deg in α, and it is representative of the others. The radial distance represents the time, and the angular variable represents α. The ratio between the higher and the lower values is of the order of 10. The minimum times belong to the regions $120 \leq \alpha \leq 180$ deg and $300 \leq \alpha \leq 360$ deg. These results lead to the most important conclusion of this section. They show that it is possible to obtain a minimum time of flight that is 10 times shorter than the maximum without any reduction in the savings because $C3$ is kept constant. The only task that has to be performed is to find the value of α that allows these savings in time. This information can be obtained from Fig. 4.

Effects of Direction of Velocity

Several simulations were made, keeping constant the mass parameter ($\mu = 0.0121285627$) and the periapse distance ($r_p = 1838$ km) and changing the values of $C3$ ($-0.2 \leq C3 \leq 0$) for posigrade and retrograde orbits. The orbit is called posigrade when the initial velocity is counterclockwise and retrograde when it is clockwise. The simulations showed that, for values of $C3$ in the first half of the interval considered ($-0.1 \leq C3 \leq 0$), in the majority of the domain (values of α) the time of flight required for the capture is almost independent of the direction of the velocity. A significant difference occurs only in very specific positions ($30 \leq \alpha \leq 70$ deg and $220 \leq \alpha \leq 260$ deg) and, in those cases, the posigrade orbits have a smaller value for the time of flight. But, for the most important cases ($C3$ about -0.2), where the savings are close to the maximum, there are significant differences in the time required for the capture for almost all of the values of α. Figure 5 shows the result for $C3 = -0.17$. It was built using a step of 1 deg in α. It is clear that the posigrade orbits require a smaller time for the capture for all values of α. The ratio of those times (time required for the retrograde orbits divided by the time required for the posigrade orbits) can reach three in the region $130 \leq \alpha \leq 180$ deg in the second plot. The posigrade (direct) orbits holds all of the minimums. When $C3$ approaches the value of -0.2, the occurrence of retrograde orbits decreases faster than the occurrence of posigrade orbits. In this situation, the posigrade orbits dominate the plots, and they are the only choice in a large portion of the domain. In the small parts of the domain that have retrograde orbits, the difference in the time for capture increases.



$C3 = -0.10$



$C3 = -0.17$

Fig. 5 Time of flight vs α for a posigrade and a retrograde orbit: ■, posigrade and •, retrograde.

Effects of C3

This is also a very important topic of investigation. To perform this research, simulations were made keeping $\mu = 0.0121285627$ and the direction of the velocity posigrade. A set of simulations was performed in the interval $0 \leq \alpha \leq 360$ deg. This study is to quantify numerically the balance that exists between consumption of fuel and time required for the maneuver. The approach to solve this problem is the following. A value of $C3$ is fixed, and then a plot of the time of flight vs α is made. From this simulation, the minimum value of the time of flight is found. Repeating this process for several values of $C3$ it is possible to build Table 1, which shows the maximum magnitude of $C3$ obtained for every value of time of flight. The values of α and ΔV saved (in canonical units) are also shown in Table 1. Figure 6 shows these results in graphic form for $r_p = 1838$ km. The expected result, that an increase in the savings

Table 1 Savings in minimum time (canonical units)

Time	α	$C3$	Savings in ΔV
0.49	309	0	0.00000
0.5	310	0.01	0.00222
0.51	311	0.02	0.00444
0.52	312	0.03	0.00667
0.53	315	0.04	0.00890
0.54	319	0.05	0.01113
0.56	314	0.06	0.01336
0.57	320	0.07	0.01559
0.59	318	0.08	0.01783
0.61	318	0.09	0.02007
0.63	319	0.1	0.02231
0.65	322	0.11	0.02455
0.68	321	0.12	0.02680
0.71	323	0.13	0.02905
0.74	327	0.14	0.03130
0.78	331	0.15	0.03355
0.84	328	0.16	0.03580
0.91	329	0.17	0.03806
1	333	0.18	0.04032
1.14	337	0.19	0.04258
1.41	340	0.2	0.04484
3.32	345	0.21	0.04711

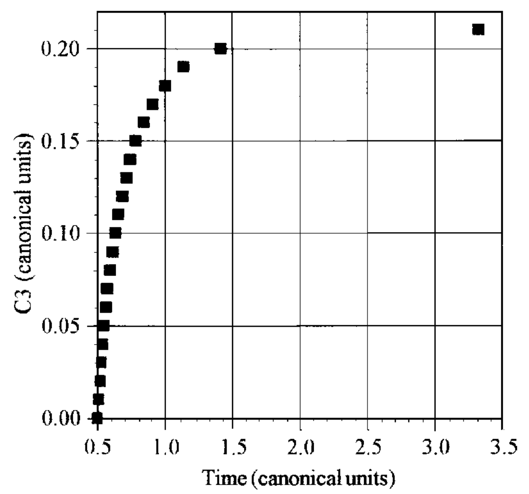


Fig. 6 Minimum time of flight vs $C3$ for r_p = 1838 km.

obtained causes an increase in the time of flight, is quantified in the plot. Several other simulations changing the values of r_p, not shown here, were made, and it was possible to conclude that r_p is a parameter that has little effect in this problem. It is also evident from Table 1 that the regions where the minimums are found are always in the region close to the interval $309 \leq \alpha \leq 345$ deg. These results can be used to solve some practical optimal problems, as explained in the next section.

Optimal Problems

Several families of optimal problems can be solved using the results obtained in this research. Two families of these problems are discussed.

1) Suppose that it is required to design a trajectory that ends in a gravitational capture for a given system of binaries and for a fixed value of r_p. Assume that these values are $\mu = 0.0121285627$ (the Earth–moon system) and $r_p = 0.004781477$ (which represents an altitude of 100 km above the surface of the moon). The requirement in this problem is that the trajectory uses the minimum amount of time possible but keeping a fixed value of $C3 = -0.14$. Figure 6 and Table 1 can be directly used to solve this problem. From Fig. 6, it is possible to see that the time of flight is 0.74 in canonical units. Table 1 shows that the value of α that generates this situation is $\alpha = 327$ deg. It is also shown that the saving in ΔV obtained is 0.031296 canonical units. Figure 7 shows the trajectory of the spacecraft, as seen from the rotating frame of reference. This type of problem can be solved for different values of μ, r_p, $C3$, etc. Similar situations with more degrees of freedom (a free value of r_p, for example) can also be solved using the same technique.

2) Another variant of the optimal problem that can be solved with the data shown is the problem of searching for a trajectory that allows the maximum savings subject to a inequality constraint in time (time required for the capture less than a specified limit). Figure 6 can solve this problem. It is possible to read the time limit in the horizontal axis and then find the value of $C3$ that is obtained in the vertical line. Suppose that the same binary system (Earth–moon) and $r_p = 0.004781477$ are used and that the limit in time is 0.8 in canonical units. Then the maximum savings that can be obtained corresponds to $C3 = -0.15$. Table 1 gives the value of $\alpha = 331$ deg. Figure 8 shows the trajectory obtained, as seen from the rotating frame.

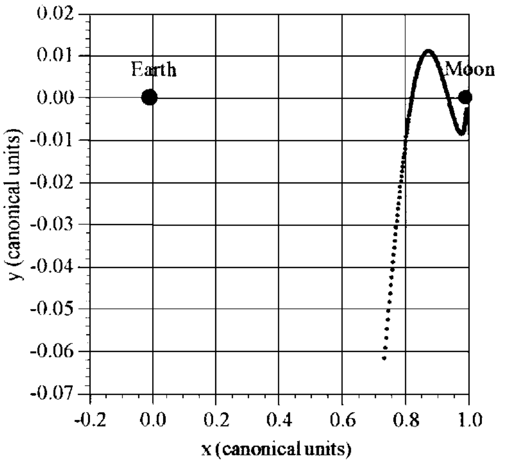


Fig. 7 Trajectory that solves first optimal problem.

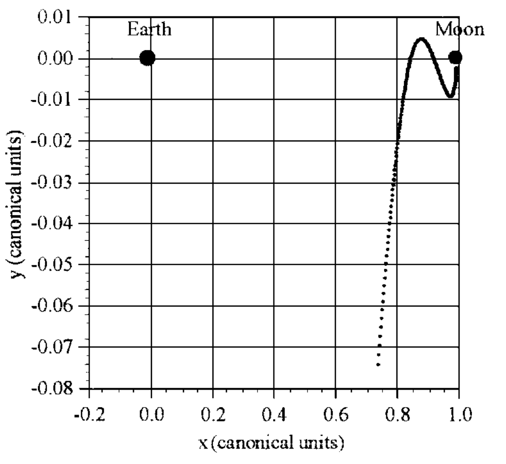


Fig. 8 Trajectory that solves second optimal problem.

Conclusion

This paper studied the gravitational capture in the regularized, restricted three-body problem. The preliminary results confirmed the results found previously in Refs. 1–3 that showed the characteristics and importance of this problem for Earth–moon transfers. A detailed, new study of the time of flight required for the gravitational capture was performed. The importance of each individual parameter was studied in detail. Windows with short time for capture that are very pronounced for values of $C3$ close to the minimum (≈ -0.2) were found. Some blank regions, where gravitational capture is not possible, were also found. Optimal problems, such as finding trajectories that end in gravitational capture with minimum time or maximum savings, are also solved using the results available in this paper. Those results are important to mission designers willing to use this type of maneuver in real missions.

Acknowledgments

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